A NUMERICAL EXAMPLE OF LINEAR MODEL CALCULATIONS

by

S. R. Searle

Biometrics Unit, Cornell University, Ithaca, New York

BU-749-M January, 1981

Abstract

Hand calculations for both fixed effects models and variance components estimation are presented in full detail for a small hypothetical set of unbalanced data in a 2-way crossed classification. The presentation is in note form, without any details of theory but with generous literature references thereto. References are also given to calculated results as obtained from several statistical computing packages.

No comments are offered on the relative advantages and disadvantages of the different methods of estimation, nor on the different results they yield. The sole purpose of these calculations is as illustration of how the methods are applied to data. The data are hypothetical, and comparative statements would have no merit.

Paper No. BU-749-M in the Biometrics Unit

TABLE OF CONTENTS

1. Data

The data set used is Data Set 3 of the Annotated Computer Output (ACO) project.

	y_{ijk}			$y_{i\cdot\cdot}$
	7, 9	6	2	24
	8	4, 8	12	32
$y_{\cdot j\cdot}$	24	18	14	$56 = y_{\cdots}$

	n_{ij}			$n_{i\cdot}$
	2	1	1	4
	1	2	1	4
$n_{\cdot j}$	3	3	2	$8 = n_{\cdot\cdot} = N$

Notation is in LM ("Linear Models", Wiley, 1971)

Model $E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$ (with interaction)

Model equations

$$
\begin{aligned}
7 &= \mu + \alpha_1 + \beta_1 + \gamma_{11} + e_{111} \\
9 &= \mu + \alpha_1 + \beta_1 + \gamma_{11} + e_{112} \\
6 &= \mu + \alpha_1 + \beta_2 + \gamma_{12} + e_{121} \\
2 &= \mu + \alpha_1 + \beta_3 + \gamma_{13} + e_{131} \\
8 &= \mu + \alpha_2 + \beta_1 + \gamma_{21} + e_{211} \\
4 &= \mu + \alpha_2 + \beta_2 + \gamma_{22} + e_{221} \\
8 &= \mu + \alpha_2 + \beta_2 + \gamma_{22} + e_{222} \\
12 &= \mu + \alpha_2 + \beta_3 + \gamma_{23} + e_{231}
\end{aligned}
$$

LM 289

Vector & matrix notation (dot = zero)

$$
y = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{131} \\ y_{211} \\ y_{221} \\ y_{222} \\ y_{231} \end{bmatrix}
= \begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix}
= \begin{bmatrix}
1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}
+ \begin{bmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{131} \\ e_{211} \\ e_{221} \\ e_{222} \\ e_{231} \end{bmatrix}
$$

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$$

Normal equations

$$
\left[\begin{array}{c|cc|ccc|cccccc}
8 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\
4 & 4 & \cdot & 2 & 1 & 1 & 2 & 1 & 1 & \cdot & \cdot & \cdot \\
4 & \cdot & 4 & 1 & 2 & 1 & \cdot & \cdot & \cdot & 1 & 2 & 1 \\
\hline
3 & 2 & 1 & 3 & \cdot & \cdot & 2 & \cdot & \cdot & 1 & \cdot & \cdot \\
3 & 1 & 2 & \cdot & 3 & \cdot & \cdot & 1 & \cdot & \cdot & 2 & \cdot \\
2 & 1 & 1 & \cdot & \cdot & 2 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\hline
2 & 2 & \cdot & 2 & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
2 & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
\begin{bmatrix} \mu^0 \\ \alpha_1^0 \\ \alpha_2^0 \\ \beta_1^0 \\ \beta_2^0 \\ \beta_3^0 \\ \gamma_{11}^0 \\ \gamma_{12}^0 \\ \gamma_{13}^0 \\ \gamma_{21}^0 \\ \gamma_{22}^0 \\ \gamma_{23}^0 \end{bmatrix}
=
\begin{bmatrix} 56 \\ 24 \\ 32 \\ 24 \\ 18 \\ 14 \\ 16 \\ 6 \\ 2 \\ 8 \\ 12 \\ 12 \end{bmatrix}
$$

LM 290

Solution vector

$$\mu^0 = 0 \qquad \beta_1^0 = 0$$
$$\alpha_1^0 = 0 \qquad \beta_2^0 = 0$$
$$\alpha_2^0 = 0 \qquad \beta_3^0 = 0$$

and then

$$\begin{bmatrix} 2 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 2 & \\ &&&&& 1 \end{bmatrix} \begin{bmatrix} \gamma_{11}^0 \\ \gamma_{12}^0 \\ \gamma_{13}^0 \\ \gamma_{21}^0 \\ \gamma_{22}^0 \\ \gamma_{23}^0 \end{bmatrix} = \begin{bmatrix} 16 \\ 6 \\ 2 \\ 8 \\ 12 \\ 12 \end{bmatrix}$$

LM 292.

$$\gamma_{11}^0 = 16/2 = 8$$
$$\gamma_{12}^0 = 6/1 = 6$$
$$\gamma_{13}^0 = 2/1 = 2$$
$$\gamma_{21}^0 = 8/1 = 8$$
$$\gamma_{22}^0 = 12/2 = 6$$
$$\gamma_{23}^0 = 12/1 = 12.$$

Solution $b^{0\prime} = [0 \mid 0\ 0 \mid 0\ 0\ 0 \mid 8\ 6\ 2\ 8\ 6\ 12]$

2. Sums of Squares

LM 392

$P(\mu) = N\bar{y}_{\cdot\cdot\cdot}^2 = y_{\cdot\cdot\cdot}^2/N = 56^2/8 = 392$

$R(\mu,\alpha) = \sum_{i=1}^{a} n_{i\cdot}\bar{y}_{i\cdot\cdot}^2 = \sum_{i=1}^{2} y_{i\cdot\cdot}^2/n_{i\cdot}$

$= 24^2/4 + 32^2/4$

$= 144 + 256 = 400$

$R(\mu,\beta) = \sum_{j=1}^{b} n_{\cdot j}\bar{y}_{\cdot j\cdot}^2 = \sum_{j=1}^{3} y_{\cdot j\cdot}^2/n_{\cdot j}$

$= 24^2/3 + 18^2/3 + 14^2/2$

$= 192 + 108 + 98 = 398$

$R(\mu,\alpha,\beta) = R(\alpha|\mu,\beta) + R(\mu,\beta)$

$= 398$, as above

$\underset{\sim}{u}'T^{-1}\underset{\sim}{u}$, from (69) + (70) of LM 297

$\underset{\sim}{u}$ in this case is scalar, + T is scalar ($\because a=2$)

$u_1 = y_{1\cdot\cdot} - \sum_{j=1}^{3} n_{1j}\bar{y}_{\cdot j\cdot} = 24 - \left[2\left(\frac{24}{3}\right) + 1\left(\frac{18}{3}\right) + 1\left(\frac{14}{2}\right)\right]$

$= 24 - (16 + 6 + 7)$

$= -5.$

$T = t_{11} = n_{1\cdot} - \sum_{j=1}^{3} n_{1j}^2/n_{\cdot j} = 4 - \left(\frac{2^2}{3} + \frac{1^2}{3} + \frac{1^2}{2}\right) = 4 - 2\frac{1}{6} = 1\frac{5}{6}$

$\therefore \quad R(\alpha | \mu, \beta) = u_1 t_{11}^{-1} u_{11} = 5^2 / 1\frac{5}{6} = \frac{150}{11} = 13\frac{7}{11}$

check In this case, with $a = 2$, we also have

$$R(\alpha | \mu, \beta) = u_2 t_{22}^{-1} u_2.$$

$$u_2 = y_{2..} - \sum_{j=1}^{3} n_{2j} \bar{y}_{.j.} = 32 - [1(\tfrac{24}{3}) + 2(\tfrac{18}{3}) + 1(\tfrac{14}{2})]$$
$$= 32 - (8 + 12 + 7)$$
$$= 5$$

$$t_{22} = n_{2.} - \sum_{j=1}^{3} n_{2j}^2 / n_{.j} = 4 - (\tfrac{1^2}{3} + \tfrac{2^2}{3} + \tfrac{1^2}{2}) = 4 - 2\tfrac{1}{6} = 1\tfrac{5}{6}$$

$$\therefore R(\alpha | \mu, \beta) = 5^2 / 1\tfrac{5}{6} = 150/11 = 13\tfrac{7}{11} \checkmark$$

We also have

$$R(\mu, \alpha, \beta, \gamma) = \underset{\sim}{b}^{0\prime} \underset{\sim}{X}' \underset{\sim}{y} \qquad (61) \text{ LM } 292$$

$$= \sum\sum n_{ij} \bar{y}_{ij.}^2 = \sum\sum y_{ij.}^2 / n_{ij}$$

$$= \frac{16^2}{2} + \frac{6^2}{1} + \frac{2^2}{1} + \frac{8^2}{1} + \frac{12^2}{2} + \frac{12^2}{1}$$
$$= 128 + 36 + 4 + 64 + 72 + 144 = 448$$

$$\text{Total s/s} = \sum\sum\sum y_{ijk}^2 = 7^2 + 9^2 + 6^2 + 2^2 + 8^2 + 4^2 + 8^2 + 12^2 = 458$$

$$SSE = SST - R(\mu, \alpha, \beta, \gamma) = 458 - 448 = 10$$
$$R(\mu) = 392$$
$$R(\alpha | \mu) = R(\mu, \alpha) - R(\mu) = 400 - 392 = 8$$
$$R(\beta | \mu) = R(\mu, \beta) - R(\mu) = 398 - 392 = 6$$
$$R(\mu, \alpha, \beta) = R(\alpha | \mu, \beta) + R(\mu, \beta) = 13\tfrac{7}{11} + 398 = 411\tfrac{7}{11}$$
$$R(\beta | \mu, \alpha) = R(\mu, \alpha, \beta) - R(\mu, \alpha) = 411\tfrac{7}{11} - 400 = 11\tfrac{7}{11}$$
$$R(\gamma | \mu, \alpha, \beta) = R(\mu, \alpha, \beta, \gamma) - R(\mu, \alpha, \beta) = 448 - 411\tfrac{7}{11} = 36\tfrac{4}{11}$$

checks

$$SSE = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})^2$$

$$= (7-8)^2 + (9-8)^2 + (6-6)^2 + (2-2)^2 + (8-8)^2 + (4-6)^2 + (8-6)^2 + (12-12)^2$$

$$= 1^2 + 1^2 + 0 + 0 + 0 + 2^2 + 2^2 + 0$$

$$= 10 \checkmark$$

$R(\beta|\mu,\alpha) = r' C^{-1} r$ based on (63), LM 293

$$\underset{\sim}{C}: \begin{bmatrix} 3-\left(\frac{2^2}{4}+\frac{1^2}{4}\right) & -\left(\frac{2(1)}{4}+\frac{1(2)}{4}\right) & -\left(\frac{2(1)}{4}+\frac{1(1)}{4}\right) \\ & 3-\left(\frac{1^2}{4}+\frac{2^2}{4}\right) & -\left(\frac{1(1)}{4}+\frac{2(1)}{4}\right) \\ \text{sym} & & 2-\left(\frac{1^2}{4}+\frac{1^2}{4}\right) \end{bmatrix} = \begin{bmatrix} 1\frac{3}{4} & -1 & -\frac{3}{4} \\ -1 & 1\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{3}{4} & 1\frac{1}{2} \end{bmatrix}$$

Row sums are zero — as they should be.

$$\underset{\sim}{r} = \begin{bmatrix} 24 - [2(6)+1(8)] \\ 18 - [1(6)+2(8)] \\ 14 - [1(6)+1(8)] \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$$

Elements add to zero — as they should.
We use r and $\underset{\sim}{C}$ of order 2:

$$\underset{\sim}{r}' \underset{\sim}{C}^{-1} \underset{\sim}{r} = [4 \quad -4] \begin{bmatrix} 1\frac{3}{4} & -1 \\ -1 & 1\frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 16 [1 \quad -1] \left\{ \frac{1}{4} \begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 64 [1 \quad -1] \frac{1}{33} \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{64}{33}(7+7-8) = \frac{128}{11}$$

$$= 11\tfrac{7}{11} = R(\beta|\mu,\alpha) \checkmark$$

3. Analysis of variance. LM 294, 298

Source	df	Sum of Squares		
Mean	1	$R(\mu)$	=	392
Rows after mean	1	$R(\alpha\|\mu)$	=	8
Columns after mean & rows	2	$R(\beta\|\mu,\alpha)$	=	$11\frac{2}{11}$
Interaction	2	$R(\gamma\|\mu,\alpha,\beta)$	=	$36\frac{4}{11}$
Error	2	SSE	=	10
Total	8	SST	=	458

Mean	1	$R(\mu)$	=	392
Columns after mean	2	$R(\beta\|\mu)$		6
Rows after mean & columns	1	$R(\alpha\|\mu,\beta)$		$13\frac{2}{11}$
Interaction	2	$R(\gamma\|\mu,\alpha,\beta)$		$36\frac{4}{11}$
Error	2	SSE		10
Total	8			458

SAS GLM with factors in sequence : rows, cols, interaction

Type I

For A	$R(\alpha\|\mu)$	=	8		8.00
For B	$R(\beta\|\mu,\alpha)$	=	$11\frac{2}{11}$	=	11.6363
For AB	$R(\gamma\|\mu,\alpha,\beta)$	=	$36\frac{4}{11}$	=	36.3636
Error	SSE	=	10	=	10.0000

Type II

For A	$R(\alpha\|\mu,\beta)$	=	$13\frac{2}{11}$	=	13.6363
For B	$R(\beta\|\mu,\alpha)$	=	$11\frac{2}{11}$	=	11.6363
For AB	$R(\gamma\|\mu,\alpha,\beta)$	=	$36\frac{4}{11}$	=	36.3636
Error	SSE	=	10	=	10.0000

4. Hypotheses

tested by F-statistics available in Analyses of Variance on page 7.

$$\hat{\sigma}^2 = SSE/[N - r(X)] \qquad LM\ 171$$
$$= 10/2$$
$$= 5.$$

1. $F(\mu) = [R(\mu)/1]5 = 392/5 = 78.4$

$H: E(\bar{y}_{..}) = 0$ (97), LM 306

$\equiv H: \mu + \frac{1}{8}(4\alpha_1 + 4\alpha_2) + \frac{1}{8}(3\beta_1 + 3\beta_2 + \beta_3)$
$+ \frac{1}{8}(2\gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{21} + 2\gamma_{22} + \gamma_{23}) = 0.$

2. $F(\alpha|\mu) = [R(\alpha|\mu)/1]5 = 8/5 = 1.6$

$H: \alpha_i + \frac{1}{n_{i.}}\sum n_{ij}(\beta_j + \gamma_{ij})$ equal $\forall i$ (100), LM 307

$\equiv H: \alpha_1 + \frac{1}{4}(2\beta_1 + \beta_2 + \beta_3 + 2\gamma_{11} + \gamma_{12} + \gamma_{13}) = \alpha_2 + \frac{1}{4}(\beta_1 + 2\beta_2 + \beta_3 + \gamma_{21} + 2\gamma_{22} + \gamma_{23})$

3. $F(\beta|\mu) = [R(\beta|\mu)/2]/5 = \frac{1}{2}(11\frac{7}{11})/5 = 1.163$

$H: \beta_j + \frac{1}{n_{.j}} \sum_i n_{ij}(\alpha_i + \gamma_{ij})$ equal $\forall i$ LM 308

$\equiv H: \beta_1 + \frac{1}{3}(2\alpha_1 + \alpha_2 + 2\gamma_{11} + \gamma_{21}) = \beta_2 + \frac{1}{3}(\alpha_1 + 2\alpha_2 + \gamma_{12} + 2\gamma_{22}) = \beta_3 + \frac{1}{2}(\alpha_1 + \alpha_2 + \gamma_{13} + \gamma_{23})$

4. $F(\alpha|\mu,\beta) = [R(\alpha|\mu,\beta)/1]/5 = 13\frac{7}{11}/5 = 2.7272$

$H: \varphi_1 = 0$ (104), LM 308

$\equiv H: \left(n_{i.} - \sum_{j=1}^{b} \frac{n_{ij}^2}{n_{.j}}\right)\alpha_i - \sum_{i' \neq i}^{a}\left(\sum_{j=1}^{b} \frac{n_{ij} n_{i'j}}{n_{.j}}\right)\alpha_{i'}$

$+ \sum_{j=1}^{b}\left(n_{ij} - \frac{n_{ij}^2}{n_{.j}}\right)\gamma_{ij} - \sum_{i' \neq i}^{a}\left(\sum_{j=1}^{b} \frac{n_{ij} n_{i'j}}{n_{.j}}\right)\gamma_{i'j} = 0$

for $i = 1$. (86), LM 30

$\equiv H: \left[4 - \left(\frac{2^2}{3} + \frac{1^2}{3} + \frac{1^2}{2}\right)\right]\alpha_1 - \left(\frac{2(1)}{3} + \frac{1(2)}{3} + \frac{1(1)}{2}\right)\alpha_2$

$+ \left(2 - \frac{2^2}{3}\right)\gamma_{11} + \left(1 - \frac{1^2}{3}\right)\gamma_{12} + \left(1 - \frac{1^2}{2}\right)\gamma_{13} - \left[\frac{2(1)}{3}\gamma_{21} + \frac{1(2)}{3}\gamma_{22} + \frac{1(1)}{2}\gamma_{23}\right] =$

$\equiv H: 1\frac{5}{6}\alpha_1 - 1\frac{5}{6}\alpha_1 + \frac{2}{3}\gamma_{11} + \frac{2}{3}\gamma_{12} + \frac{1}{2}\gamma_{13} - \frac{2}{3}\gamma_{21} - \frac{2}{3}\gamma_{22} - \frac{1}{2}\gamma_{23} = 0$

5. $F(\beta|\mu,\alpha) = [R(\beta|\mu,\alpha)/2]/5 = 11\tfrac{2}{11}/10 = 1.1636$

$$H: \left(n_{.j} - \sum_i \frac{n_{ij}^2}{n_{i.}}\right)\beta_j - \sum_{j'\neq j}\left(\sum_{i=1}^{a} \frac{n_{ij}n_{ij'}}{n_{i.}}\right)\beta_{j'}$$ (106) LM 309

$$+ \sum_{i=1}^{a}\left(n_{ij} - \frac{n_{ij}^2}{n_{i.}}\right)\gamma_{ij} - \sum_{j'\neq j}\sum_{i=1}^{a} \frac{n_{ij}n_{ij'}}{n_{i.}}\gamma_{ij'} = 0 \quad \text{for } j = 1,2.$$

$$\equiv H: \begin{cases} \left[3 - \left(\frac{2^2}{4} + \frac{1^2}{4}\right)\right]\beta_1 - \left[\frac{2(1)}{4} + \frac{1(2)}{4}\right]\beta_2 - \left[\frac{2(1)}{4} + \frac{1(1)}{4}\right]\beta_3 \\ \quad + \left(2 - \frac{2^2}{4}\right)\gamma_{11} + \left(1 - \frac{1^2}{4}\right)\gamma_{21} - \left[\frac{2(1)}{4}\gamma_{12} + \frac{2(1)}{4}\gamma_{13} + \frac{1(2)}{4}\gamma_{22} + \frac{1(1)}{4}\gamma_{23}\right] = 0 \\ \left[3 - \left(\frac{1^2}{4} + \frac{2^2}{4}\right)\right]\beta_2 - \left[\frac{1(2)}{4} + \frac{2(1)}{4}\right]\beta_1 - \left[\frac{1(1)}{4} + \frac{1(2)}{4}\right]\beta_3 \\ \quad + 1 - \frac{1^2}{4})\gamma_{12} + \left(2 - \frac{2^2}{4}\right)\gamma_{22} - \left[\frac{1(2)}{4}\gamma_{11} + \frac{2(1)}{4}\gamma_{21} + \frac{1(1)}{4}\gamma_{13} + \frac{2(1)}{4}\gamma_{23}\right] = 0 \end{cases}$$

$$\equiv H: \begin{cases} 1\tfrac{3}{4}\beta_1 - \beta_2 - \tfrac{3}{4}\beta_3 + \gamma_{11} + \tfrac{3}{4}\gamma_{21} - \tfrac{1}{2}\gamma_{12} - \tfrac{1}{2}\gamma_{13} - \tfrac{1}{2}\gamma_{22} - \tfrac{1}{4}\gamma_{23} = 0 \\ -\beta_1 \quad 1\tfrac{3}{4}\beta_2 - \tfrac{3}{4}\beta_3 + \tfrac{3}{4}\gamma_{12} + \gamma_{22} - \tfrac{1}{2}\gamma_{11} - \tfrac{1}{2}\gamma_{21} - \tfrac{1}{4}\gamma_{13} - \tfrac{1}{2}\gamma_{23} = 0. \end{cases}$$

6. $F(\gamma|\mu,\alpha,\beta) = [R(\gamma|\mu,\alpha,\beta)/2]/5 = 3.63636$

$$H: \begin{cases} \mu_{11} - \mu_{12} - \mu_{21} - \mu_{22} = 0 \\ \mu_{12} - \mu_{13} - \mu_{22} + \mu_{23} = 0. \end{cases}$$

$$\equiv H: \begin{cases} \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0 \\ \gamma_{12} - \gamma_{13} - \gamma_{22} + \gamma_{23} = 0 \end{cases}$$

Check calculation for $R(\gamma|\mu, \alpha, \beta)$.

Numerator s/s for $H: L'\gamma = 0$ is

$$Q = \bar{y}' L \left[L' D\{\tfrac{1}{n_{ij}}\} L \right]^{-1} L' \bar{y} \qquad (93),\ LM\ 305$$

$$H: \quad \begin{aligned} \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} &= 0 \\ \gamma_{12} - \gamma_{13} - \gamma_{22} + \gamma_{23} &= 0 \end{aligned}$$

$$\equiv H: \; L'\gamma = 0 \quad \text{with} \quad L' = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\bar{y}' = \begin{bmatrix} 8 & 6 & 2 & 8 & 6 & 12 \end{bmatrix}$$

$$D\{\tfrac{1}{n_{ij}}\} = \begin{bmatrix} \tfrac{1}{2} & 1 & 1 & 1 & \tfrac{1}{2} & 1 \end{bmatrix}$$

$$L'\bar{y} = \begin{bmatrix} 8-6-8+6 \\ 6-2-6+12 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\left[L' D\{\tfrac{1}{n_{ij}}\} L \right]^{-1} = \begin{bmatrix} \tfrac{1}{2}+1+1+\tfrac{1}{2} & -1-\tfrac{1}{2} \\ -1-\tfrac{1}{2} & 1+1+\tfrac{1}{2}+1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1\tfrac{1}{2} \\ -1\tfrac{1}{2} & 3\tfrac{1}{2} \end{bmatrix}^{-1} = \frac{1}{10\tfrac{1}{2} - 2\tfrac{1}{4}} \begin{bmatrix} 3\tfrac{1}{2} & 1\tfrac{1}{2} \\ 1\tfrac{1}{2} & 3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 10 \end{bmatrix} \frac{1}{8\tfrac{1}{4}} \begin{bmatrix} 3\tfrac{1}{2} & 1\tfrac{1}{2} \\ 1\tfrac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \frac{300}{8\tfrac{1}{4}} = \frac{400}{11} = 36\tfrac{4}{11} = 36.3636 \checkmark$$

Σ - restricted Models

Reference

SSH: Searle S.R., Speed F.M. & Henderson H.V., "Some computational and model equivalences in analyses of variance of unequal-subclass-numbers data" *The American Statistician* 35, Feb 1981.

Restrictions (A10) SSH

$$
\left.\begin{array}{r}
\dot\alpha_1 + \dot\alpha_2 = 0 \\
\dot\beta_1 + \dot\beta_2 + \dot\beta_3 = 0 \\
\dot\gamma_{11} + \dot\gamma_{12} + \dot\gamma_{13} = 0 \\
\dot\gamma_{21} + \dot\gamma_{22} + \dot\gamma_{23} = 0 \\
\dot\gamma_{11} + \dot\gamma_{21} = 0 \\
\dot\gamma_{12} + \dot\gamma_{22} = 0 \\
\dot\gamma_{13} + \dot\gamma_{23} = 0
\end{array}\right\}
\Rightarrow
\begin{array}{l}
\dot\alpha_2 = -\dot\alpha_1 \\
\dot\beta_3 = -\dot\beta_1 - \dot\beta_2 \\
\dot\gamma_{11} = \dot\gamma_{11} \\
\dot\gamma_{12} = \dot\gamma_{12} \\
\dot\gamma_{13} = -\dot\gamma_{11} - \dot\gamma_{12} \\
\dot\gamma_{21} = -\dot\gamma_{11} \\
\dot\gamma_{22} = -\dot\gamma_{12} \\
\dot\gamma_{23} = \dot\gamma_{11} + \dot\gamma_{12}
\end{array}
$$

Model equations

$$
\begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdot & 1 & \cdot \\
1 & 1 & 1 & \cdot & 1 & \cdot \\
1 & 1 & \cdot & 1 & \cdot & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & \cdot & 1 & \cdot & 1 & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & 1 \\
1 & \cdot & \cdot & 1 & \cdot & 1 \\
1 & \cdot & -1 & -1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} \dot\mu \\ \dot\alpha_1 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\gamma_{11} \\ \dot\gamma_{12} \end{bmatrix}
+ \underset{\sim}{e}
$$

Normal Equations (A12), SSH

$$\begin{bmatrix} 8 & 0 & 1 & 1 & 1 & -1 \\ 0 & 8 & 1 & -1 & 1 & 1 \\ 1 & 1 & 5 & 2 & 1 & 0 \\ 1 & -1 & 2 & 5 & 0 & -1 \\ 1 & 1 & 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\alpha}_1 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\gamma}_{11} \\ \dot{\gamma}_{12} \end{bmatrix} = \begin{bmatrix} 56 \\ -8 \\ 10 \\ 4 \\ 18 \\ 4 \end{bmatrix}$$

Solution to normal equations (A14), SSH

$$\begin{bmatrix} \dot{\mu} \\ \dot{\alpha}_1 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\gamma}_{11} \\ \dot{\gamma}_{12} \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 10 & 0 & -1 & -1 & -3 & 3 \\ 0 & 10 & -3 & 3 & -1 & -1 \\ -1 & -3 & 19 & -8 & -3 & 0 \\ -1 & 3 & -8 & 19 & 0 & 3 \\ -3 & -1 & -3 & 0 & 19 & -8 \\ 3 & -1 & 0 & 3 & -8 & 19 \end{bmatrix} \begin{bmatrix} 56 \\ -8 \\ 10 \\ 4 \\ 18 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -\frac{10}{6} \\ 1 \\ -1 \\ \frac{10}{6} \\ \frac{10}{6} \end{bmatrix}$$

Check on $R(\mu, \alpha, \beta, \gamma)$

$R(\mu, \alpha, \beta, \gamma)$ = inner product of sol. and r.h.s

$$= 56(7) + (-8)(-10/6) + 10(1) + 4(-1) + 18(10/6) + 4(10/6)$$

$$= 392 + 6 + \tfrac{10}{6}(8 + 18 + 4)$$

$$= 448 \checkmark$$

6 Sums of Squares in Restricted Models

Restrictions can be used simply as a convenience for getting full-rank formulation of non full rank models.

Restrictions can also be used to define a different model from an unrestricted model, & within that redefined model to then consider sums of squares.

$$R^*(\dot{\alpha} \mid \dot{\mu}, \dot{\beta}, \dot{\gamma})_\Sigma = R^*(\dot{\mu}, \dot{\alpha}, \dot{\beta}, \dot{\gamma})_\Sigma - R^*(\dot{\mu}, \dot{\beta}, \dot{\gamma})_\Sigma$$

$$\boxed{= R(\mu, \alpha, \beta, \gamma) - R^*(\dot{\mu}, \dot{\beta}, \dot{\gamma})_\Sigma}$$

This equality holds because $\dot{\mu}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}$ is just a reparametrization of $\mu, \alpha, \beta, \gamma$.

This is not $R(\mu, \beta, \gamma)$ because $\dot{\mu}\ \dot{\beta}\ \dot{\gamma}$ is not a reparameterization of μ, β, γ. It is only part of a reparameterization of $\mu, \alpha, \beta, \gamma$.

See SSH (8.14) and surrounding text.

$R^*(\mu\ \beta\ \gamma)_\Sigma$ – obtained by deleting α's and α-equations from normal equations for $\mu\ \alpha\ \beta$ & γ's. From page 13 this gives

$$\begin{bmatrix} 8 & 1 & 1 & 1 & -1 \\ 1 & 5 & 2 & 1 & 0 \\ 1 & 2 & 5 & 0 & -1 \\ 1 & 1 & 0 & 5 & 2 \\ -1 & 0 & -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} \hat\mu \\ \hat\beta_1 \\ \hat\beta_2 \\ \hat\gamma_{11} \\ \hat\gamma_{12} \end{bmatrix} = \begin{bmatrix} 56 \\ 10 \\ 4 \\ 18 \\ 4 \end{bmatrix} \text{ with solution } \begin{bmatrix} 7 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1\frac{1}{2} \\ 1\frac{1}{2} \end{bmatrix} \quad \text{(A15), SSH}$$

$$R^*(\mu, \beta, \gamma)_\Sigma = 56(7) + 10(\tfrac{1}{2}) + 4(-\tfrac{1}{2}) + 18(1\tfrac{1}{2}) + 4(1\tfrac{1}{2})$$
$$= 392 + 5 - 2 + 27 + 6$$
$$= 428$$

$$R^*(\alpha \mid \mu, \beta\ \gamma)_\Sigma = 448 - 428 = 20.$$

Similarly delete β's & β-equations from p. 13

$$\begin{bmatrix} 8 & 0 & 1 & -1 \\ 0 & 8 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ -1 & 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} \hat\mu \\ \hat\alpha_1 \\ \hat\gamma_{11} \\ \hat\gamma_{12} \end{bmatrix} = \begin{bmatrix} 56 \\ -8 \\ 18 \\ 4 \end{bmatrix} \text{ with solution } \begin{bmatrix} 7 \\ -\frac{13}{9} \\ \frac{16}{9} \\ \frac{16}{9} \end{bmatrix}$$

$$R^*(\mu, \alpha, \gamma)_\Sigma = 56(7) + (-8)(-13/9) + 22(16/9) = 392 + \tfrac{152}{3} = 442\tfrac{2}{3}$$

$$R^*(\beta \mid \mu\ \alpha\ \gamma)_\Sigma = 448 - 442\tfrac{2}{3} = 5\tfrac{1}{3}.$$

7 Weighted Squares of Means Analysis

LM 370

For all-cells-filled cases

$R^*(\dot\alpha|\dot\mu, \dot\beta, \dot\gamma)_\Sigma = SSA_w$ } from weighted squares of means
$R^*(\dot\beta|\dot\mu\ \dot\alpha\ \dot\gamma)_\Sigma = SSB_w$ }

means

8	6	2	16/3
8	6	12	26/3
$\frac{16}{2}$	$\frac{12}{2}$	$\frac{14}{2}$	

n_{ij}'s

2	1	1	4
1	2	1	4
3	3	2	8

LM 370: For rows

$\frac{1}{w_1} = \frac{1}{9}\left(\frac{1}{2}+\frac{1}{1}+\frac{1}{1}\right) = \frac{5}{18}$

$\frac{1}{w_2} = \frac{1}{9}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{1}\right) = \frac{5}{18}$

$\bar{x}_{[1]} = \left(\frac{18}{5}\ \frac{16}{3} + \frac{18}{5}\ \frac{26}{3}\right) / 2\left(\frac{18}{5}\right)$

$= 7$

$SSA_w = \frac{18}{5}\left(\frac{16}{3}-7\right)^2 + \frac{18}{5}\left(\frac{26}{3}-7\right)^2$

$= \frac{18}{5}\ \frac{1}{9}\left[(-5)^2 + (5)^2\right]$

$= 20$

$= R^*(\dot\alpha|\dot\mu\ \dot\beta\ \dot\gamma)_\Sigma$, p.15

For columns

$\frac{1}{v_1} = \frac{1}{4}\left(\frac{1}{2}+\frac{1}{1}\right) = \frac{3}{8}$

$\frac{1}{v_2} = \frac{1}{4}\left(\frac{1}{1}+\frac{1}{2}\right) = \frac{3}{8}$

$\frac{1}{v_3} = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{1}\right) = 1$

$\bar{x}_{[2]} = \left[\frac{8}{3}(8) + \frac{8}{3}(6) + 7\right] / \left(\frac{8}{3}+\frac{8}{3}+1\right)$

$= 44\frac{1}{3} / \frac{19}{3} = 7$

$SSB_w = \frac{8}{3}(8-7)^2 + \frac{8}{3}(6-7)^2 + 1(7-7)^2$

$= \frac{16}{3}$

$= 5\frac{1}{3}$

$= R^*(\dot\beta|\dot\mu\ \dot\alpha\ \dot\gamma)_\Sigma$, p.15

SAS GLM type III — for all-cells-filled cases

For A : $SSA_w = R^*(\dot{\alpha} \mid \dot{\mu}, \dot{\beta}, \dot{\gamma})_\Sigma = 20$

Tests : $H: \alpha_1 + \frac{1}{3}(\gamma_{11} + \gamma_{12} + \gamma_{13}) = \alpha_2 + \frac{1}{3}(\gamma_{21} + \gamma_{22} + \gamma_{23})$

For B : $SSB_w = R^*(\dot{\beta} \mid \dot{\mu}, \dot{\alpha}, \dot{\gamma})_\Sigma = 5\frac{1}{3}$

Tests $H: \beta_1 + \frac{1}{2}(\gamma_{11} + \gamma_{21}) = \beta_2 + \frac{1}{2}(\gamma_{12} + \gamma_{22}) = \beta_3 + \frac{1}{2}(\gamma_{13} + \gamma_{23})$

For AB $R(\gamma \mid \mu, \alpha, \beta) = 36.3636$

Tests : $\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0$

$\gamma_{12} - \gamma_{13} - \gamma_{22} + \gamma_{23} = 0.$

8. Different sums of squares for the mean

(1) $R(\mu) = N\bar{y}^2 = 8(49) = 392$

(2) $R^*(\dot\mu \mid \dot\alpha\ \dot\beta\ \dot\gamma)_\Sigma = R^*(\dot\mu\ \dot\alpha\ \dot\beta\ \dot\gamma)_\Sigma - R^*(\dot\alpha\ \dot\beta\ \dot\gamma)_\Sigma$

From p. 13 the normal equations without $\dot\mu$ are

$$\begin{bmatrix} 8 & 1 & -1 & 1 & 1 \\ 1 & 5 & 2 & 1 & 0 \\ -1 & 2 & 5 & 0 & -1 \\ 1 & 1 & 0 & 5 & 2 \\ 1 & 0 & -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} \hat{\dot\alpha} \\ \hat{\dot\beta}_1 \\ \hat{\dot\beta}_2 \\ \hat{\dot\gamma}_{11} \\ \hat{\dot\gamma}_{12} \end{bmatrix} = \begin{bmatrix} -8 \\ 10 \\ 4 \\ 18 \\ 4 \end{bmatrix} \text{ with solution } \frac{1}{30}\begin{bmatrix} -50 \\ 51 \\ -9 \\ 113 \\ -13 \end{bmatrix}$$

Therefore

$$R^*(\dot\alpha\ \dot\beta\ \dot\gamma)_\Sigma = \tfrac{1}{30}\left[-8(-50) + 10(51) + 4(-9) + 18(113) + 4(-13)\right]$$
$$= 95.2$$

$$R^*(\dot\mu \mid \dot\alpha\ \dot\beta\ \dot\gamma)_\Sigma = 448 - 95.2 = 352.8$$

(3) For the no interaction case the normal equations for the Σ-restricted model, with $\dot\mu$ deleted are (use top of p. 13)

$$\begin{bmatrix} 8 & 1 & -1 \\ 1 & 5 & 2 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} \hat{\dot\alpha}_1 \\ \hat{\dot\beta} \\ \hat{\dot\beta}_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 10 \\ 4 \end{bmatrix} \text{ with solution } \frac{1}{11}\begin{bmatrix} -15 \\ 27 \\ -5 \end{bmatrix}$$

$$R^*(\dot\alpha, \dot\beta)_\Sigma = \tfrac{1}{11}\left[-8(-15) + 10(27) + 4(-5)\right] = \tfrac{370}{11} = 33\tfrac{7}{11}$$
$$R^*(\dot\mu \mid \dot\alpha, \dot\beta)_\Sigma = R(\dot\mu\ \dot\alpha\ \dot\beta)_\Sigma - R^*(\dot\alpha, \dot\beta) = 411\tfrac{1}{11} - 33\tfrac{7}{11} = 378.$$

7. An "indirect" calculation for full rank models

$$Q_1 = \hat{b}_1' T_{11}^{-1} \hat{b}_1 \quad \text{— see (2.6), SSH.}$$

Examples: using normal equations & solutions, p. 13.

(1) $R^*(\mu \mid \dot\alpha, \dot\beta, \dot\gamma)_\Sigma = 7\left(\frac{10}{72}\right)^{-1} 7 = \frac{49(72)}{10} = 352.8$ ✓, see p.18

(2) $R^*(\dot\alpha \mid \mu, \dot\beta, \dot\gamma)_\Sigma = -\frac{10}{6}\left(\frac{10}{72}\right)^{-1}\left(-\frac{10}{6}\right) = \frac{100}{36}\,\frac{72}{10} = 20$ ✓, see pp. 15-16

(3) $R^*(\dot\beta \mid \mu, \dot\alpha, \dot\gamma)_\Sigma = \begin{bmatrix} 1 & -1 \end{bmatrix} \left[\frac{1}{72}\begin{pmatrix} 19 & -8 \\ -8 & 19 \end{pmatrix} \right]^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= \frac{72}{11(27)} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 19 & 8 \\ 8 & 19 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$= \frac{8}{33}\left[2(19) - 2(8)\right] = \frac{16}{3} = 5\tfrac{1}{3}$ ✓, see pp 15-16.

VARIANCE COMPONENTS.

RANDOM MODEL

2	1	1	4
1	2	1	4
3	3	2	

10 Henderson Method 1

Re-arranging first 5 lines of LM 481

$$E\,R(\alpha|\mu) = E(T_A - T_\mu)$$
$$E\,R(\beta|\mu) = E(T_B - T_\mu)$$

$$E\,R(\alpha|\mu) = (N-k_1')\sigma_\alpha^2 + (k_3 - k_2')\sigma_\beta^2 + (k_3 - k_{23}')\sigma_\gamma^2 + (a-1)\sigma_e^2$$

$$E\,R(\beta|\mu) = (k_4 - k_1')\sigma_\alpha^2 + (N - k_2')\sigma_\beta^2 + (k_4 - k_{23}')\sigma_\gamma^2 + (b-1)\sigma_e^2$$

$$E(T_{AB} - T_A - T_B + T_\mu) = (k_1' - k_4)\sigma_\alpha^2 + (k_2' - k_3)\sigma_\beta^2 + (N - k_3 - k_4 + k_{23}') + (s-a-b+1)\sigma_e^2$$

$$E(T_0 - T_{AB}) = E(SSE) = (N-s)\sigma_e^2$$

From LM 480, $N = 8$

$k_1 = 4^2 + 4^2 = 32$ $\qquad k_1' = 32/8 = 4$

$k_2 = 3^2 + 3^2 + 2^2 = 22$ $\qquad k_2' = 22/8 = 2\frac{3}{4}$

$k_3 = \dfrac{2^2+1^2+1^2}{4} + \dfrac{1^2+2^2+1^2}{4} = 3$

$k_4 = \dfrac{2^2+1^2}{3} + \dfrac{1^2+2^2}{3} + \dfrac{1^2+1^2}{2} = 4\frac{1}{3}$

$k_{23} = 2^2+1^2+1^2+1^2+2^2+1^2 = 12$ $\qquad k_{23}' = 12/8 = 1\frac{1}{2}$

$$E\ R(\alpha|\mu) = 4\sigma_\alpha^2 + \tfrac{1}{4}\sigma_\beta^2 + 1\tfrac{1}{2}\sigma_\gamma^2 + \sigma_e^2$$

$$E\ R(\beta|\mu) = \tfrac{1}{3}\sigma_\alpha^2 + 5\tfrac{1}{4}\sigma_\beta^2 + 2\tfrac{5}{6}\sigma_\gamma^2 + 2\sigma_e^2$$

$$E\ (T_{AB} - T_A - T_B + T_\mu) = -\tfrac{1}{3}\sigma_\alpha^2 - \tfrac{1}{4}\sigma_\beta^2 + 2\tfrac{1}{6}\sigma_\gamma^2 + 2\sigma_e^2$$

$$E\ (SSE) = 2\sigma_e^2$$

Henderson I estimates

From p. 7. $R(\alpha|\mu) = 8$
$R(\beta|\mu) = 6$

And $T_{AB} - T_A - T_B + T_\mu = \sum\sum n_{ij}\bar{y}_{ij}^2 - \sum n_i\bar{y}_{i\cdot}^2 - \sum n_j\bar{y}_{\cdot j}^2 + N\bar{y}_{\cdot\cdot}^2$

$= R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha) - R(\mu,\beta) + N\bar{y}^2$

$= 448 - 400 - 398 + 392$, from pp 4-5

$= 42.$

$$8 = 4\hat{\sigma}_\alpha^2 + \tfrac{1}{4}\hat{\sigma}_\beta^2 + 1\tfrac{1}{2}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$$
$$6 = \tfrac{1}{3}\hat{\sigma}_\alpha^2 + 5\tfrac{1}{4}\hat{\sigma}_\beta^2 + 2\tfrac{5}{6}\hat{\sigma}_\gamma^2 + 2\hat{\sigma}_e^2$$
$$42 = -\tfrac{1}{3}\hat{\sigma}_\alpha^2 - \tfrac{1}{4}\hat{\sigma}_\beta^2 + 2\tfrac{1}{6}\hat{\sigma}_\gamma^2 + 2\hat{\sigma}_e^2$$
$$10 = 2\hat{\sigma}_e^2$$

$\hat{\sigma}^2 = 5$

$$4\hat{\sigma}_\alpha^2 + \tfrac{1}{4}\hat{\sigma}_\beta^2 + 1\tfrac{1}{2}\hat{\sigma}_\gamma^2 = 3$$
$$\tfrac{1}{3}\hat{\sigma}_\alpha^2 + 5\tfrac{1}{4}\hat{\sigma}_\beta^2 + 2\tfrac{5}{6}\hat{\sigma}_\gamma^2 = -4$$
$$-\tfrac{1}{3}\hat{\sigma}_\alpha^2 - \tfrac{1}{4}\hat{\sigma}_\beta^2 + 2\tfrac{1}{6}\hat{\sigma}_\gamma^2 = 32$$

$$-2\tfrac{3}{4}\hat{\sigma}_\beta^2 + 27\tfrac{1}{2}\hat{\sigma}_\gamma^2 = 387$$
$$5\hat{\sigma}_\beta^2 + 5\hat{\sigma}_\gamma^2 = 28$$
$$\hat{\sigma}_\beta^2 + \hat{\sigma}_\gamma^2 = 5.6$$
$$30\tfrac{1}{4}\hat{\sigma}_\gamma^2 = 387 + 2\tfrac{3}{4}(5.6) = 387 + 11.2 + 4.2$$
$$= 402.4$$
$$\hat{\sigma}_\gamma^2 = 1609.6/121 = 13.30247$$
$$\hat{\sigma}_\beta^2 = 5.6 - 1609.6/121$$
$$= -932/121 = -7.7024$$
$$\hat{\sigma}_\alpha^2 = \tfrac{1}{4}\left[3 + \tfrac{1}{4}(932)/121 - 1\tfrac{1}{2}(1609.6)/121\right]$$
$$= \tfrac{1}{4}\left[3 + \frac{233 - 2414.4}{121}\right]$$
$$= -\frac{454.6}{121} = -1.6909$$

Check from LM 481

$$\delta_A = [445 - 400 - 4(5)]/5 = \frac{28}{5}$$

$$\delta_B = [448 - 398 - 3(5)]/3\tfrac{2}{3} = \frac{105}{11}$$

$$\hat{\sigma}_\gamma^2 = \frac{4\left(\frac{105}{11}\right) + \frac{1}{4}\left(\frac{28}{5}\right) - \{(400 - 392) - 5\}}{8 - 4 - 2\frac{3}{4} + 1\frac{1}{2}}$$

$$= \frac{\frac{420}{11} + \frac{7}{5} - 3}{2\frac{3}{4}} = \frac{2100 - 88}{55\,(11/4)} = \frac{8048}{5(121)}$$

$$= \frac{1609.6}{121} \checkmark$$

$$\hat{\sigma}_\beta^2 = \delta_A - \hat{\sigma}_\gamma^2 = \frac{28}{5} - \frac{1609.6}{121} = -\frac{[1609.6 - 121(5.6)]}{121}$$

$$= -\frac{932}{121} \checkmark$$

$$\hat{\sigma}_\alpha^2 = \delta_B - \hat{\sigma}_\gamma^2 = \frac{105}{11} - \frac{1609.6}{121} = -\frac{(1609.6 - 1155)}{121}$$

$$= -\frac{454.6}{121} \checkmark$$

II. Nonlinear Method 3

Can be used in any one of 3 sub-methods.

Sub-method.	Sums of Squares used.	Source in LM: Table	Source in LM: Equivalent equations at bottom of LM488	
(i)	$R(\alpha\|\mu)$ $R(\beta\|\mu,\alpha)$ $R(\gamma\|\mu,\alpha,\beta)$ SSE	10.1 LM447	2nd + 3rd	SAS GLM Type I when factors entered in sequence α, β, γ.
(ii)	$R(\beta\|\mu)$ $R(\alpha\|\mu,\beta)$ $R(\gamma\|\mu,\alpha,\beta)$ SSE	10.2 LM448	1st + 3rd	
(iii)	$R(\alpha\|\mu,\beta)$ $R(\beta\|\mu,\alpha)$ $R(\gamma\|\mu,\alpha,\beta)$ SSE	–	1st + 2nd	SAS GLM Type II

Sub-method i)

$$\left.\begin{array}{l} E\ R(\alpha|\mu) \\ E\ SSE \end{array}\right\} \text{ from } p\ 21$$

From the middle two lines of Table 10.7, LM 447

$$E\ R(\beta|\mu,\alpha) = h_4\sigma_\beta^2 + (h_4 - h_6)\sigma_\gamma^2 + (b-1)\sigma_e^2$$

$$E\ R(\gamma|\mu,\alpha,\beta) = h_6\sigma_\gamma^2 + (s-a-b+1)\sigma_e^2$$

The difficult calculation is h_6.
From Table 11.3, LM 484

$$\lambda_1 = \tfrac{1}{4}(4+1+1) = 1\tfrac{1}{2} \qquad \lambda_2 = \tfrac{1}{4}(4+1+1) = 1\tfrac{1}{2}$$

There is a symmetric matrix F_i for the i'th row of the data. Initially it is calculated of order b to check that row sums are zero. Then it is used in reduced form, of order b-1.

$$F_1 = \begin{bmatrix} \frac{1}{4}(2^2)(1\frac{1}{2} + 4 - 4) & \frac{1}{4}(2)(1\frac{1}{2} - 2 - 1) & \frac{1}{4}(2)(1\frac{1}{2} - 2 - 1) \\ & \frac{1}{4}(1^2)(1\frac{1}{2} + 4 - 2) & \frac{1}{4}(1\frac{1}{2} - 1 - 1) \\ \text{(sym)} & & \frac{1}{4}(1^2)(1\frac{1}{2} + 4 - 2) \end{bmatrix}$$

$$= \begin{bmatrix} 1\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{7}{8} & -\frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} & \frac{7}{8} \end{bmatrix}$$

$$F_2 = \begin{bmatrix} \frac{1}{4}(1^2)(1\frac{1}{2} + 4 - 2) & \frac{1}{4}(2)(1\frac{1}{2} - 1 - 2) & \frac{1}{4}(1)(1\frac{1}{2} - 1 - 1) \\ & \frac{1}{4}(2^2)(1\frac{1}{2} + 4 - 4) & \frac{1}{4}(2)(1\frac{1}{2} - 2 - 1) \\ \text{(sym)} & & \frac{1}{4}(1^2)(1\frac{1}{2} + 4 - 2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{8} & -\frac{3}{4} & -\frac{1}{8} \\ -\frac{3}{4} & 1\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{8} & -\frac{3}{4} & \frac{7}{8} \end{bmatrix}$$

In reduced form

$$F_1 = \frac{1}{8}\begin{bmatrix} 12 & -6 \\ -6 & 7 \end{bmatrix} \qquad F_2 = \frac{1}{8}\begin{bmatrix} 7 & -6 \\ -6 & 12 \end{bmatrix} \qquad F_1 + F_2 = \frac{1}{8}\begin{bmatrix} 19 & -12 \\ -12 & 19 \end{bmatrix}$$

and from p. 6 we use the reduced C

$$C = \begin{bmatrix} 1\frac{3}{4} & -1 \\ -1 & 1\frac{3}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix} \quad \text{with} \quad \underset{\sim}{C}^{-1} = \frac{4}{33}\begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}.$$

Then, from Table 11.3, LM 484

$$k^* = \sum_i \lambda_i + tr\left(C^{-1} \sum_i F_i\right)$$

$$= 1\tfrac{1}{2} + 1\tfrac{1}{2} + tr\left\{ \frac{4}{33}\begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix} \frac{1}{8}\begin{bmatrix} 19 & -12 \\ -12 & 19 \end{bmatrix}\right\}$$

$$= 3 + 2\left[7(19) + 4(-12)\right]/66$$

$$= 5\tfrac{19}{33}$$

Hence

$$h_6 = N - k^* = 8 - 5\tfrac{19}{33} = 2\tfrac{14}{33} = 2.4242.$$

Therefore with

$$h_4 = N - k_3, \text{ from LM 483}$$
$$= 8 - 3, \text{ from p. 20}$$
$$= 5$$

p. 25 gives

$$E\,R(\beta|\mu,\alpha) = 5\sigma_\beta^2 + 2\tfrac{14}{33}\sigma_\gamma^2 + 2\sigma_e^2$$

$$E\,R(\gamma|\mu,\alpha,\beta) = 2\tfrac{14}{33}\sigma_\gamma^2 + 2\sigma_e^2.$$

Using Vise, p 7 and p 21

Sums of squares

$R(\alpha\mid\mu)$	1	8	$= 4\hat{\sigma}_\alpha^2 + \frac{1}{4}\hat{\sigma}_\beta^2 + 1\frac{1}{2}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$
$R(\beta\mid\mu,\alpha)$	2	$11\frac{7}{11}$	$= 5\hat{\sigma}_\beta^2 + 2\frac{19}{33}\hat{\sigma}_\gamma^2 + 2\hat{\sigma}_e^2$
$R(\gamma\mid\mu,\alpha,\beta)$	2	$36\frac{4}{11}$	$= 2\frac{14}{33}\hat{\sigma}_\gamma^2 + 2\hat{\sigma}_e^2$
SSE	2	10	$= 2\hat{\sigma}_e^2$

Mean squares

$$8 = 8.0000 = 4\hat{\sigma}_\alpha^2 + \frac{1}{4}\hat{\sigma}_\beta^2 + 1\frac{1}{2}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$$

$$5\frac{9}{11} = 5.8181 \qquad 2\frac{1}{2}\hat{\sigma}_\beta^2 \quad 1.2878\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$$

$$18\frac{2}{11} = 18.1818 \qquad 1.2121\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$$

$$5 = 5.0000 \qquad \hat{\sigma}_e^2$$

Estimates

$$\hat{\sigma}_e^2 = 5 \qquad = 5$$

$$\hat{\sigma}_\gamma^2 = \frac{18\frac{2}{11} - 5}{1\frac{7}{33}} = \frac{87}{8} = 10\frac{7}{8} \qquad = 10.875$$

$$\hat{\sigma}_\beta^2 = \frac{5\frac{9}{11} - 5 - 1\frac{19}{66}(10\frac{7}{8})}{2\frac{1}{2}} = -5\frac{11}{40} \qquad = -5.275$$

$$\hat{\sigma}_\alpha^2 = \frac{1}{4}\left[8 - 5 - 1\frac{1}{2}(10\frac{7}{8}) - \frac{1}{4}(-5\frac{11}{40})\right] = -2\frac{639}{640} = -2.9984$$

See ACO σ^2: SAS VARCOMP p XIV
ACO σ^2: SAS RANDOM p. IX Type I.

Sub-method (ii) This uses

$$
\begin{aligned}
E\,R(\alpha|\mu,\beta) &= E\,R(\alpha|\mu) + E\,R(\beta|\mu,\alpha) - E\,R(\beta|\mu) \\
&= 4\sigma_\alpha^2 + \tfrac{1}{4}\sigma_\beta^2 + 1\tfrac{1}{2}\sigma_\gamma^2 + \sigma_e^2 \\
&\quad + 5\sigma_\beta^2 + 2\tfrac{19}{33}\sigma_\gamma^2 + 2\sigma_e^2 \\
&\quad - \left(\tfrac{1}{3}\sigma_\alpha^2 + 5\tfrac{1}{4}\sigma_\beta^2 + 2\tfrac{5}{6}\sigma_\gamma^2 + 2\sigma_e^2\right) \\
&= 3\tfrac{2}{3}\sigma_\alpha^2 + 1\tfrac{8}{33}\sigma_\gamma^2 + \sigma_e^2
\end{aligned}
$$

Sums of squares

$R(\beta\|\mu)$	2	6	$= \tfrac{1}{3}\hat\sigma_\alpha^2 + 5\tfrac{1}{4}\hat\sigma_\beta^2 + 2\tfrac{5}{6}\hat\sigma_\gamma^2 + 2\hat\sigma_e^2$
$R(\alpha\|\mu,\beta)$	1	$13\tfrac{7}{11}$	$= 3\tfrac{2}{3}\hat\sigma_\alpha^2 + 1\tfrac{8}{33}\hat\sigma_\gamma^2 + \hat\sigma_e^2$
$R(\gamma\|\mu,\beta,\gamma)$	2	$36\tfrac{4}{11}$	$2\tfrac{14}{33}\sigma_\gamma^2 + 2\hat\sigma_e^2$
SSE	2	10	$2\hat\sigma_e^2$

Mean squares

3	$= 3.0000$	$= \tfrac{1}{6}\hat\sigma_\alpha^2 + 2\tfrac{5}{8}\sigma_\beta^2 + 1\tfrac{5}{12}\hat\sigma_\gamma^2 + \hat\sigma_e^2$
$13\tfrac{7}{11}$	$= 13.6363$	$= 3\tfrac{2}{3}\hat\sigma_\alpha^2 + 1\tfrac{8}{33}\hat\sigma_\gamma^2 + \hat\sigma_e^2$
$18\tfrac{2}{11}$	$= 18.1818$	$= 1\tfrac{7}{33}\sigma_\gamma^2 + \hat\sigma_e^2$
5	$= 5.000$	$= \hat\sigma_e^2$

1.2424

1.2121

Estimates

$\hat\sigma_e^2 = 5$ $= 5$

$\hat\sigma_\gamma^2 = 10\tfrac{7}{8}$, as on p.28 $= 10.875$

$$\hat\sigma_\alpha^2 = \frac{13\tfrac{7}{11} - 5 - 1\tfrac{8}{33}\left(10\tfrac{7}{8}\right)}{3\tfrac{2}{3}} = -\frac{117}{88} = -1.3295$$

$$\hat\sigma_\beta^2 = \frac{3 - 5 - 1\tfrac{5}{12}\left(10\tfrac{7}{8}\right) - \tfrac{1}{6}\left(-\tfrac{117}{88}\right)}{2\tfrac{5}{8}} = -\frac{6049}{924} = -6.5465$$

$=$

Sub-method (iii)

Mean squares: from p 28 & 29

from $R(\alpha \mid \mu, \beta)$	$13\frac{2}{11}$	=	$3\frac{2}{3}\hat{\sigma}_\alpha^2 + 1\frac{8}{33}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$
$R(\beta \mid \mu, \alpha)$	$5\frac{9}{11}$	=	$2\frac{1}{2}\hat{\sigma}_\beta^2 + 1\frac{19}{66}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$
$R(\gamma \mid \mu, \alpha, \beta)$	$18\frac{2}{11}$	=	$2\frac{14}{33}\hat{\sigma}_\gamma^2 + \hat{\sigma}_e^2$
SSE	$\cancel{2}5$	=	$\hat{\sigma}_e^2$

See ACO σ^2 SAS RANDOM p. IX Type II

Using $\hat{\sigma}_\beta^2$ from p. 28 and $\hat{\sigma}_\alpha^2$ from p. 29

$$\hat{\sigma}_e^2 = 5$$
$$\hat{\sigma}_\gamma^2 = 10.575$$
$$\hat{\sigma}_\beta^2 = -5.275$$
$$\hat{\sigma}_\alpha^2 = -1.3295$$

12. Without Interaction

Henderson Method 1.

From p. 21

$$8 = 4\hat{\sigma}_\alpha^2 + \tfrac{1}{4}\hat{\sigma}_\beta^2 + \hat{\sigma}_e^2$$
$$6 = \tfrac{1}{3}\hat{\sigma}_\alpha^2 + 5\tfrac{1}{4}\hat{\sigma}_\beta^2 + 2\hat{\sigma}_e^2$$
$$52 = -\tfrac{1}{3}\hat{\sigma}_\alpha^2 - \tfrac{1}{4}\hat{\sigma}_\beta^2 + 4\hat{\sigma}_e^2$$

$$60 = 3\tfrac{2}{3}\hat{\sigma}_\alpha^2 + 5\hat{\sigma}_e^2$$
$$58 = 5\hat{\sigma}_\beta^2 + 6\hat{\sigma}_e^2$$

$$8 = 4\left[\frac{60-5\hat{\sigma}_e^2}{3\frac{2}{3}}\right] + \frac{1}{4}\left[\frac{58-6\hat{\sigma}_e^2}{5}\right] + \hat{\sigma}_e^2$$

$$\hat{\sigma}_e^2 = \frac{8 - \frac{240}{3\frac{2}{3}} - \frac{58}{20}}{1 - \frac{20}{3\frac{2}{3}} - \frac{6}{20}} = \frac{\frac{720}{11} + \frac{29}{10} - 8}{\frac{60}{11} + \frac{3}{10} - 1}$$

$$= \frac{7200 + 319 - 880}{600 + 33 - 110} = \frac{6639}{523} \qquad \hat{\sigma}_e^2 = 12.6941$$

$$\hat{\sigma}_\alpha^2 = \frac{60 - \frac{5(6639)}{523}}{3\frac{2}{3}} = \frac{-5445}{5753} \qquad \hat{\sigma}_\alpha = -.946463$$

$$\hat{\sigma}_\beta^2 = \frac{58 - \frac{6(6639)}{523}}{5} = \frac{-1900}{523} \qquad \hat{\sigma}_{\cancel{\beta}}^2 = -3.63288$$

Henderson Method 3

- Sub-method (i) - from p. 28

$$R(\alpha|\mu) \quad 1 \quad 8 = 4\hat\sigma_\alpha^2 + \tfrac{1}{4}\hat\sigma_\beta^2 + \hat\sigma_e^2$$

$$R(\beta|\mu,\alpha) \quad 2 \quad 11\tfrac{7}{11} = 5\hat\sigma_\beta^2 + 2\hat\sigma_e^2$$

$$SSE \quad 4 \quad 46\tfrac{4}{11} = 4\hat\sigma_e^2$$

Mean squares:

$$8 = 4\hat\sigma_\alpha^2 + \tfrac{1}{4}\hat\sigma_\beta^2 + \hat\sigma_e^2$$

$$5\tfrac{9}{11} = 2\tfrac{1}{2}\hat\sigma_\beta^2 + \hat\sigma_e^2$$

$$11\tfrac{13}{22} = \hat\sigma_e^2$$

Estimates

$$\hat\sigma_e^2 = 11\tfrac{13}{22} = 11.5909$$

$$\hat\sigma_\beta^2 = \frac{5\tfrac{9}{11} - 11\tfrac{13}{22}}{2\tfrac{1}{2}} = -2\tfrac{17}{55} = -2.3091$$

$$\hat\sigma_\alpha^2 = \tfrac{1}{4}\left[8 - 11\tfrac{13}{22} - \tfrac{1}{4}\left(-2\tfrac{17}{55}\right)\right] = -\frac{663}{880} = -.7534$$

Sub-method (ii) — from p. 29

$R(\beta \mid \mu)$	2	6	=	$\frac{1}{3}\hat{\sigma}_\alpha^2 + 5\frac{1}{4}\hat{\sigma}_\beta^2 + 2\hat{\sigma}_e^2$
$R(\alpha \mid \mu, \beta)$	1	$13\frac{7}{11}$	=	$3\frac{2}{3}\hat{\sigma}_\alpha^2 + \hat{\sigma}_e^2$
SSE	4	$46\frac{4}{11}$	=	$4\hat{\sigma}_e^2$

Mean Squares

$$3 = \frac{1}{6}\sigma_\alpha^2 + 2\frac{5}{8}\hat{\sigma}_\beta^2 + \hat{\sigma}_e^2$$

$$13\frac{7}{11} = 3\frac{2}{3}\hat{\sigma}_\alpha^2 + \hat{\sigma}_e^2$$

$$11\frac{13}{22} = \hat{\sigma}_e^2$$

Estimates

$$\hat{\sigma}_e^2 = 11\frac{13}{22} \qquad 11\ 5909$$

$$\hat{\sigma}_\alpha^2 = \left(13\frac{7}{11} - 11\frac{13}{22}\right)\Big/3\frac{2}{3} = \frac{135}{242} = .5578$$

$$\sigma_\beta^2 = \frac{3 - 11\frac{13}{22} - \frac{1}{6}\left(\frac{135}{242}\right.}{2\frac{5}{8}}$$

$$= -\frac{2802}{847} \qquad \doteq -3.3081$$

Sub-method (iii)

From (i) & (ii)	$\hat{\sigma}_e^2 = 11\frac{13}{22}$	ACO σ^2 SAS HARVEY p. VII
From (i)	$\hat{\sigma}_\beta^2 = -2.3091$	
From (ii)	$\hat{\sigma}_\alpha^2 = .5578$	

13 MINQUE(U) = MINQUE(0)

In general $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\alpha} + \sum_{i=1}^{c+1} Z_i \underset{\sim}{b}_i$

fixed effects

$\underbrace{b_1 \ldots b_c}$ random effects

$b_{c+1} = \underset{\sim}{e}$

$Z_{c+1} = I_N$

MINQUE(0) is for $M = I - X(X'X)^- X'$

$$\{tr(\underset{\sim}{M}\underset{\sim}{Z}_i\underset{\sim}{Z}_i'\underset{\sim}{M}\underset{\sim}{Z}_j\underset{\sim}{Z}_j')\}\,\hat{\underset{\sim}{\sigma}}^2 = \{\underset{\sim}{y}'\underset{\sim}{M}\underset{\sim}{Z}_i\underset{\sim}{Z}_i'\underset{\sim}{M}\underset{\sim}{y}\}_{i,j=1,\ldots,c+1}$$

which is equivalent to

$$\{ssqe(Z_i' M Z_j)\}\,\hat{\underset{\sim}{\sigma}}^2 = \{ssqe(Z_i' M y)\}$$

where $ssqe(\underset{\sim}{A}) \equiv$ sum of squares of elements of $\underset{\sim}{A}$.

[See ACO σ^2 SAS VARCOMP p. 3].

For random model, μ is only fixed effect. Using p. 2

$$\underset{\sim}{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ Z_1 = \begin{bmatrix} 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \end{bmatrix},\ Z_2 = \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix},\ Z_3 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}\ Z_4 = I_8 .$$

$$M = I - \bar{J}_8 = I - \tfrac{1}{8} J_8$$

$y' = [7 \quad 9 \quad 6 \quad 2 \quad 8 \quad 4 \quad 8 \quad 12]$

$(My)' = [0 \quad 2 \quad -1 \quad -5 \quad 1 \quad -3 \quad 1 \quad 5]$

We calculate ssqe $(Z_i' M Z_j)$ & ssqe $(Z_i' M y)$

<u>Matrices</u> <u>s.s.q.e – as needed</u>

$Z_1'M = Z_1' - \frac{1}{2}\begin{bmatrix} 1' \\ 1' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1\;1\;1\;1 & -1\;-1\;-1\;-1 \\ -1\;-1\;-1\;-1 & 1\;1\;1\;1 \end{bmatrix}$

$Z_1'MZ_1 = 2(2I_2 - J_2)$ $\qquad 4[2(1^2) + 2(\cdot 1)^2] = 16$

$Z_1'MZ_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ $\qquad 4\left(\frac{1}{4}\right) = 1$

$Z_1'MZ_3 = \begin{bmatrix} 2\;1\;1 & \\ & 1\;2\;1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2\;1\;1 & 1\;2\;1 \\ 2\;1\;1 & 1\;2\;1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$ $\qquad 4 + 8\left(\frac{1}{4}\right) = 6$

$Z_1'MZ_4 = Z_1'M$ $\qquad \frac{1}{4}(16) = 4$

$(Z_1'My)' = (-4 \quad 4)$ $\qquad 2(4^2) = 16$

$Z_2'M = Z_2' - \frac{1}{8}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} 1_8'$

$Z_2'MZ_2 = \text{diag}\{3 \; 3 \; 2\} - \frac{1}{8}\begin{bmatrix} 9 & 9 & 6 \\ 9 & 9 & 6 \\ 6 & 6 & 4 \end{bmatrix}$ $\qquad \frac{3^2}{8^2}[2(5^2) + 4^2 + 4(3^2) + 4(2^2)] = \frac{225}{16} = 14.0625$

$$Z_2' M Z_3 = \begin{bmatrix} 2 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 6 & 3 & 3 & 3 & 6 & 3 \\ 6 & 3 & 3 & 3 & 6 & 3 \\ 4 & 2 & 2 & 2 & 4 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 10 & -3 & -3 & 5 & -6 & -3 \\ -6 & 5 & -3 & -3 & 10 & -3 \\ -4 & -2 & 6 & -2 & -4 & 6 \end{bmatrix}$$

$$\frac{1}{64}\left[2(2^2) + 2(4^2) + 4(6^2) + 2(5^2) + 6(3^2) + 2(10^2)\right] = \frac{244}{32} = \frac{61}{8} = 7.625$$

$$Z_2' M Z_4 = Z_2' M$$

$$= \frac{1}{8}\begin{bmatrix} 5 & 5 & -3 & -3 & 5 & -3 & -3 & -3 \\ -3 & -3 & 5 & -3 & -3 & 5 & 5 & -3 \\ -2 & -2 & -2 & 6 & -2 & -2 & -2 & 6 \end{bmatrix}$$

$$\frac{1}{64}\left[6(5^2) + 10(3^2) + 6(2^2) + 2(6^2)\right] = \frac{336}{64} = 5.25$$

$$(Z_2' M y)' = [3 \quad -3 \quad 0] \qquad 2(9) = 18$$

$$Z_3' M = Z_3' - \frac{1}{8}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} 1_8'$$

$$Z_3' M Z_3 = \text{diag}\{2 \ 1 \ 1 \ 1 \ 2 \ 1\} - \frac{1}{8}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}[2 \ 1 \ 1 \ 1 \ 2 \ 1]$$

$$2\left(2 - \frac{4}{8}\right)^2 + 4\left(1 - \frac{1}{8}\right)^2 + \frac{1}{64}\left[12(1^2) + 2(4^2) + 16(2^2)\right] = \frac{37}{4} = 9.25$$

$$Z_3' M Z_4 = Z_3' M$$

$$8\left\{\frac{1}{64}\left[2(2^2) + 4(1^2)\right] + 4\left[\left(1-\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \left(1-\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2\right]\right. = 6\tfrac{1}{2} = 6.5$$

$(Z_3' My)' = [2 \;\; -1 \;\; -5 \;\; 1 \;\; -2 \;\; 5]$ $\qquad 2(1+4+25) = 60$

$Z_4' M = M$ $\qquad 8(1-\frac{1}{8})^2 + 56(\frac{1}{8})^2 = 7$

$(Z_4' My)' = (My)'$
$= [0 \;\; 2 \;\; -1 \;\; -5 \;\; 1 \;\; -3 \;\; 1 \;\; 5]$ $\qquad 3(1^2)+2^2+3^2+2(5^2) = 66$

Therefore the MIVQUEO equations are

$$\begin{bmatrix} 16 & 1 & 6 & 4 \\ & 14.0625 & 7.625 & 5.25 \\ \text{sym} & & 9.25 & 6.5 \\ & & & 7 \end{bmatrix} \begin{bmatrix} \sigma_\alpha^2 \\ \sigma_\beta^2 \\ \sigma_\gamma^2 \\ \sigma_e^2 \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ 60 \\ 66 \end{bmatrix}$$

as in ACO σ^2 SAS VARCOMP, p. XV (top).

VARIANCE COMPONENTS

Mixed model – rows fixed.

14 Henderson method 1

Cannot be used for mixed models. (LM 429)

15. Henderson method 2.

Reference: HSS : Henderson C.R., Searle S.R. and Schaeffer, L.R. The invariance and calculation of method 2 for estimating variance components. Biometrics 30, 583-588, 1974

Recall Henderson method 2 demands no interactions between fixed & random effects. (LM 442)

Model $$y_{ijk} = \mu + \underset{\substack{\uparrow \\ \text{fixed}}}{\alpha_i} + \underset{\substack{\uparrow \\ \text{random}}}{\beta_j} + e_{ijk}$$

For (1) of HSS

$$X_f = \begin{bmatrix} 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \end{bmatrix} \quad \underset{\sim}{X_r} = \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \quad y = \begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix} \quad [F_2 \vdots R_1] = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{bmatrix}$$

$[F_2 \vdots R_1]$ has rank $6 = r[1 \; X_f \; X_r]$

$$\begin{bmatrix} F_2'F_2 & F_2'R_1 \\ R_1'F_2 & R_1'R_1 \end{bmatrix}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \qquad (6) \text{ of HSS}$$

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 3 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}^{-1} = \frac{1}{33}\begin{bmatrix} 18 & -6 & -12 & -9 \\ -6 & 13 & 4 & 3 \\ -12 & 4 & 19 & 6 \\ -9 & 3 & 6 & 21 \end{bmatrix}$$

$$Q_{11} = \frac{18}{33} = \frac{6}{11} \qquad Q_{12} = \frac{1}{33}\begin{bmatrix} -6 & -12 & -9 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -2 & -4 & -3 \end{bmatrix}$$

$$\beta_f^\circ = \begin{bmatrix} \alpha_1^\circ \\ \alpha_2^\circ \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 0 & & 0 & \\ 6 & -2 & -4 & -3 \end{bmatrix}\begin{bmatrix} 32 \\ 24 \\ 18 \\ 14 \end{bmatrix} = \begin{bmatrix} 0 \\ 30/11 \end{bmatrix}$$

$$\gamma = \underset{\sim}{y} - X_f\beta_f^\circ = y - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 30/11 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 5\frac{3}{11} \\ \frac{3}{11} \\ 5\frac{3}{11} \\ 9\frac{3}{11} \end{bmatrix}$$

(17) HSS

Check $\quad S = Q_{11}F_2' + Q_{12}R_1' = \begin{bmatrix} 0\,0\,0\,0\,1\,1\,1\,1 \end{bmatrix}$

$$+ \frac{1}{11}\begin{bmatrix} -2 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$= \frac{6}{11}\begin{bmatrix} 0\,0\,0\,0\,1\,1\,1\,1 \end{bmatrix} + \frac{1}{11}\begin{bmatrix} -2 & -2 & -4 & -3 & -2 & -4 & -4 & - \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -2 & -2 & -4 & -3 & 4 & 2 & 2 & 3 \end{bmatrix}$$

$SF_2 = 1 = I_{1\times 1}$ ✓ $\qquad SR_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ ✓ $\qquad$ (18), HSS

7, 9	6	2
$5\frac{3}{11}$	$\frac{3}{11}, 5\frac{3}{11}$	$9\frac{3}{11}$

$21\frac{3}{11}$ $\quad 11\frac{6}{11}$ $\quad 11\frac{3}{11}$ $\quad 44\frac{1}{11}$

$$R(\beta|\mu) = \frac{1}{3}\left[(21\tfrac{3}{11})^2 + (11\tfrac{6}{11})^2\right] + \frac{1}{2}(11\tfrac{3}{11})^2 - \frac{1}{8}(44\tfrac{1}{11})^2$$

$$= \frac{1}{3}\left[\begin{array}{l} 441 + 121 \\ \frac{1}{11}(126 + 132) \\ \frac{1}{121}(9 + 36) \end{array}\right] + \frac{1}{2}\left(121 + \frac{66}{11} + \frac{9}{121}\right) - \frac{1}{8}\left(1936 + \frac{88}{11} + \frac{1}{121}\right)$$

$$= \frac{1}{6}\left[\begin{array}{l} 2(562) + 3(121) \\ + \frac{1}{11}\{2(258) + 3(66)\} \\ + \frac{1}{121}\{2(45) + 3(9)\} \end{array}\right] - \frac{1}{8}\left[1944 + \frac{1}{121}\right]$$

$$= \frac{1}{6}\left[1487 + \frac{714}{11} + \frac{117}{121}\right] - 243 \mp \frac{1}{8(121)}$$

$$= \frac{1}{6}\left[1487 + 64 + \frac{227}{121}\right] - 243\frac{1}{968}$$

$$= \frac{1}{6}\left(1552\frac{115}{121}\right) - 243\frac{1}{968} = 258\frac{295}{363} - 243\frac{1}{968}$$

$$= 15 + \frac{1}{121}\left(\frac{295}{3} - \frac{1}{8}\right) = 15\frac{2357}{24(121)} = 15.81163$$

$$SSE = 49 + 81 + 36 + 4 + 2(5\tfrac{3}{11}) + (\tfrac{3}{11})^2 + (9\tfrac{3}{11})^2 - 258\frac{295}{363}$$

$$= 170 + 50 + 81 + \frac{4(9)}{121} + \frac{1}{11}(60 + 54) - 258\frac{295}{363}$$

$$= 401 + \frac{114}{11} + \frac{36}{121} - 258\frac{295}{363}$$

$$= 411\frac{80}{121} - 258\frac{295}{363} = 152\frac{603 - 295}{363} = 152\frac{308}{363} = 152\frac{28}{33}$$

$R(\mu)$	1	$243\frac{1}{968}$	$= 243.001033$
$R(\beta\mid\mu)$	2	$15\frac{2357}{24(121)}$	15.811639
SSE	5	$152\frac{308}{363}$	152.848485
	8	$411\frac{80}{121}$	411.661157

From p. 21, & p 587 HSS. $E\,R(\beta\mid\mu) = 5\frac{1}{4}\sigma_\beta^2 + k_\beta\sigma_e^2$

$E\,SSE = k_e\sigma_e^2$

Thm 23 of HSS

$$k_\beta = \delta_\beta + b - 1 = \delta_\beta + 2$$
$$k_e = \delta_e + 5.$$

$$\delta_\beta = tr(A_\beta W) \qquad \delta_e = tr(A_e W)$$

where, from (25) of HSS

$$W = F_2 Q_{11} F_2' = \begin{bmatrix} 0\\0\\0\\0\\1\\1\\1\\1 \end{bmatrix} \frac{6}{11}[0000\ 1111] = \frac{6}{11}\begin{bmatrix} O_{4\times4} & O_{4\times4} \\ O_{4\times4} & J_4 \end{bmatrix}$$

& A_β is defined by

$$R(\beta\mid\mu) = \sum n_j \bar{y}_{\cdot j}^2 - N\bar{y}_{\cdot\cdot}^2 = \underset{\sim}{y}' A_\beta \underset{\sim}{y}$$

$$= \underset{\sim}{y}'\left\{\begin{bmatrix} 1/3 & 1/3 & \cdot & \cdot & 1/3 & \cdot & \cdot & \cdot \\ 1/3 & 1/3 & \cdot & \cdot & 1/3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/3 & \cdot & \cdot & 1/3 & 1/3 & \cdot \\ \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & 1/2 \\ 1/3 & 1/3 & \cdot & \cdot & 1/3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/3 & \cdot & \cdot & 1/3 & 1/3 & \cdot \\ \cdot & \cdot & 1/3 & \cdot & \cdot & 1/3 & 1/3 & \cdot \\ \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & 1/2 \end{bmatrix} - \frac{1}{8}J_8\right\}\underset{\sim}{y}$$

and A_e is defined by

$$SSE = y' A_e y = \Sigma\Sigma\Sigma y_{ijk}^2 - \Sigma n_j \bar{y}_{j\cdot}^2$$

$$= y'[I - (A_\beta + \bar{J}_8)]y$$

In both cases A and W are symmetric.

$$\therefore \ tr(AW) = \Sigma\Sigma a_{ij} w_{ij}$$

$$= \tfrac{6}{11}\Sigma(\text{elements in lower right } 4\times4 \text{ corner of } A),$$

because of the nature of W (see p 41).

$$\therefore \ \delta_\beta = tr(A_\beta W) = \tfrac{6}{11}\left[5(\tfrac{1}{3}) + \tfrac{1}{2} - 16(\tfrac{1}{8})\right] = \tfrac{1}{11}$$

$$\delta_e = tr(A_e W) = \tfrac{6}{11}\left[4(1) - 5\tfrac{1}{3} - \tfrac{1}{2}\right] = 1$$

Estimation

$$15.8116 = 5\tfrac{1}{4}\hat{\sigma}_\beta^2 + (\tfrac{1}{11}+2)\hat{\sigma}_e^2$$

$$152.8485 = (1+5)\hat{\sigma}_e^2$$

$$\hat{\sigma}_e^2 = \frac{152.8485}{6} = 25.4747$$

$$\hat{\sigma}_\beta^2 = -\left[2\tfrac{1}{11}(25.4747) - 15.8116\right]/5\tfrac{1}{4} = -7.13399$$

16. Henderson Method 3

α's are fixed.
Therefore use $R(\beta|\mu,\alpha)$
$R(\gamma|\mu,\alpha,\beta)$.

$$R(\gamma|\mu,\alpha,\beta) = R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha,\beta)$$

$$\begin{aligned} R(\beta|\mu,\alpha) &= R(\mu,\alpha,\beta) - R(\mu,\alpha) \\ &= R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha) - [R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha,\beta)] \end{aligned}$$

By (79), LM 445 these expressions contain no α's.
From p. 30

$$\begin{aligned} 5\tfrac{9}{11} &= 2\tfrac{1}{2}\hat{\sigma}^2_\beta + 1\tfrac{19}{66}\hat{\sigma}^2_\gamma + \hat{\sigma}^2_e \\ 18\tfrac{2}{11} &= 2\tfrac{14}{33}\hat{\sigma}^2_\gamma + \hat{\sigma}^2_e \\ 5 &= \hat{\sigma}^2_e \end{aligned}$$

$$\left.\begin{aligned} \hat{\sigma}^2_e &= 5 \\ \hat{\sigma}^2_\gamma &= 10.875 \\ \hat{\sigma}^2_\beta &= -5.275 \end{aligned}\right\} \text{ ACO } \sigma^2 \text{ SAS VARCOMP p XIV}$$

17. MINQUE(0) – MIXED MODEL

$$X = \begin{bmatrix} 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ 1 & \cdot & 1 \\ 1 & \cdot & 1 \\ 1 & \cdot & 1 \\ 1 & \cdot & 1 \end{bmatrix} \qquad Z_1 = \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \qquad Z_2 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad Z_3 = I_8$$

$$(X'X) = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 4 \end{pmatrix} \qquad (X'X)^{-} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} I - \bar{J}_4 & 0 \\ 0 & I - \bar{J}_4 \end{bmatrix} \qquad \bar{J}_4 = \frac{1}{4} J_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$(My)' = \begin{bmatrix} 1 & 3 & 0 & -4 & 0 & -4 & 0 & 4 \end{bmatrix}$$

<u>Matrices</u> $\qquad\qquad$ nqe (matrix)

$$Z_1'M = Z_1' - Z_1'\begin{bmatrix} \bar{J}_4 & 0 \\ 0 & \bar{J}_4 \end{bmatrix}$$

$$= Z_1' - \frac{1}{4}\begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Z_1'MZ_1 = \begin{bmatrix} 3 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 5 & 4 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 7 & -4 & -3 \\ -4 & 7 & -3 \\ -3 & -3 & 6 \end{bmatrix}$$

$$\frac{1}{16}\left[2(7^2) + 2(4^2) + 4(3^2) + 6^2\right] = \frac{101}{8} = 12.625 \checkmark$$

$$Z_1'MZ_2 = \begin{bmatrix} 2 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 4 & 2 & 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 & 4 & 2 \\ 2 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 & 3 & -2 & -1 \\ -2 & 3 & -1 & -2 & 4 & -2 \\ -2 & -1 & 3 & -1 & -2 & 3 \end{bmatrix}$$

$$\frac{1}{16}\left[4(1^2) + 8(2^2) + 4(3^2) + 2(4^2)\right] = \frac{104}{16} = \frac{13}{2} = 6.5$$

$$Z_1'MZ_3 = Z_1'M = \frac{1}{4}\begin{bmatrix} 2 & 2 & -2 & -2 & 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -2 & 2 & 2 & -2 \\ -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\frac{1}{16}\left[12(1^2) + 8(2^2) + 4(3^2)\right] = \frac{80}{16} = 5$$

$$(Z_1'My)' = [4 \quad -4 \quad 0] \qquad 2(4^2) = 32$$

$$Z_2'M = Z_2' - \frac{1}{4}\begin{bmatrix} 2\,2\,2\,2 & \\ 1\,1\,1\,1 & \\ 1\,1\,1\,1 & \\ & 1\,1\,1\,1 \\ & 2\,2\,2\,2 \\ & 1\,1\,1\,1 \end{bmatrix}$$

$$Z_2'MZ_2 = \begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 4 & 2 & 2 & & & \\ 2 & 1 & 1 & & & \\ 2 & 1 & 1 & & & \\ & & & 1 & 2 & 1 \\ & & & 4 & 7 & 2 \\ & & & 1 & 2 & 1 \end{bmatrix}$$

$$\frac{1}{16}\left[8(1^2) + 8(2^2) + 2(4^2)\right] = 6.5$$

$$Z_2'MZ_3 = Z_2'M = \frac{1}{4}\begin{bmatrix} 2 & 2 & -2 & -2 & & & & \\ -1 & 3 & -1 & -1 & & & & \\ -1 & -1 & -1 & 3 & & & & \\ & & & & 3 & -1 & -1 & -1 \\ & & & & -2 & 2 & -2 & -2 \\ & & & & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\frac{1}{16}\left[12(1^2) + 8(2^2) + 4(3^2)\right] = 5$$

$$(Z_2'My)' = [4 \quad 0 \quad -4 \quad 0 \quad -4 \quad 4] \qquad 4(4^2) = 64$$

$$2\left[4\left(1-\tfrac{1}{4}\right)^2 + 12\left(\tfrac{1}{4}\right)^2\right] = 6$$

$$Z_3' M Z_3 = M$$

$$1^2+3^2+3(4^2) = 58$$

$$(Z_4' My)' = (My)' = [1\ 3\ 0\ {-4}\ 0\ {-4}\ 0\ 4]$$

... the MINQUE(0) equations are

$$\begin{bmatrix} 12.625 & 6.5 & 5 \\ & 6.5 & 5 \\ \text{sym} & & 6 \end{bmatrix} \begin{bmatrix} \hat\sigma_\beta^2 \\ \hat\sigma_\gamma^2 \\ \hat\sigma_e^2 \end{bmatrix} = \begin{bmatrix} 32 \\ 64 \\ 58 \end{bmatrix}$$

$$\begin{bmatrix} \hat\sigma_\beta^2 \\ \hat\sigma_\gamma^2 \\ \hat\sigma_e^2 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 32 & -32 & 0 \\ -32 & 116 & -70 \\ 0 & -70 & 91 \end{bmatrix} \begin{bmatrix} 8 \\ 16 \\ 14\tfrac{1}{2} \end{bmatrix} = \frac{1}{49}\begin{bmatrix} -256 \\ 585 \\ 199\tfrac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -5.224489 \\ 11.938775 \\ 4.071428 \end{bmatrix}$$

as in ACO σ^2 SAS VARCOMP p. xv